

Characterization of non-constant lower bound of Ricci curvature via entropy inequality on Wasserstein space

Jinghai Shao ^{a)} and Bo Wu ^{b)}

^{a)} School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China

^{b)} School of Mathematical Sciences, Fudan University, Shanghai 200433, China

E-mail: shaojh@bnu.edu.cn, wubo@fudan.edu.cn

Abstract

When the Ricci curvature of a Riemannian manifold is not lower bounded by a constant, but lower bounded by a continuous function, we give a new characterization of this lower bound through the convexity of relative entropy on the probability space over the Riemannian manifold. Hence, we generalize K.T. Sturm and von Renesse's result (Comm. Pure Appl. Math. 2005) to the case with non-constant lower bound of Ricci curvature.

Keyword: Wasserstein space, Ricci curvature, optimal transport map

1 Introduction

In the work [9], K.T. Sturm and von Renesse give a characterization of lower bound of Ricci curvature of a Riemannian manifold M through the convexity property of relative entropy on the probability space over M . There they considered the setting that Ricci curvature of the manifold is lower bounded by a constant. The goal of this paper is to extend their characterization to the case that the Ricci curvature is bounded below by a continuous function. The idea of K.T. Sturm and von Renesse's [9] has been extended in [7] [11] [12] to metric measure space, and this new definition of lower bound of Ricci curvature owns the stability under the convergence of metric measure space. We refer the reader to the book [13] for more related works on this widely studied topic.

Let M be a smooth connected Riemannian manifold of dimension n . Fix a point $o \in M$ throughout this work. Denote by $\rho(x, y)$ the Riemannian distance between $x, y \in M$, and set $\rho_o(x) = \rho(o, x)$ for $x \in M$. Let $m(dx)$ be the Riemannian volume measure on M . Let $\mathcal{P}(M)$ be the set of all probability measures on M . For $\mu, \nu \in \mathcal{P}(M)$, the L^2 -Wasserstein distance between them is defined by

$$W_2(\mu, \nu) = \inf \left\{ \int_{M \times M} \rho(x, y)^2 \pi(dx, dy); \pi \in \mathcal{C}(\mu, \nu) \right\}^{1/2}$$

where $\mathcal{C}(\mu, \nu)$ stands for the set of all couplings to μ and ν , that is, the set of all probability measures π on $M \times M$ with $\pi(A \times M) = \mu(A)$ and $\pi(M \times A) = \nu(A)$ for every Borel set $A \subset M$. $\mathcal{P}(M)$ endowed with the metric $W_2(\cdot, \cdot)$ is called a Wasserstein space. The properties of $(\mathcal{P}(M), W_2)$ are closely related to the properties of M . For example, except the discussion on lower bound of Ricci curvature such as in [9] and present work, [5] showed the heat flow on M can be obtained as the gradient flow on $\mathcal{P}(M)$ for the relative entropy. This idea has been extended to deal with other kinds of differential equations in [1].

Let $\mathcal{P}_2(M)$ be the set of all probability measures μ on M which is absolutely continuous with respect to (w.r.t.) volume measure and satisfies $\int_M \rho(o, y)^2 \mu(dy) < \infty$. This work strongly depends on the optimal transport map on manifolds. In order to introduce our main result, we recall some basic results on the optimal transport map on Riemannian manifold. By [8], for any pair of absolutely continuous probability measures μ_0 and μ_1 in $\mathcal{P}_2(M)$, there exists a unique map $F : M \rightarrow M$ such that $\mu_1 = (F)_* \mu_0 := \mu_0 \circ F^{-1}$. Moreover, there exists a function ϕ such that F can be expressed in the form $F(x) = \exp_x(\nabla \phi(x))$. Let $F_t(x) = \exp_x(t \nabla \phi(x))$ for $t \in [0, 1]$. Then $\mu_t := (F_t)_* \mu_0$ is the unique geodesic in $(\mathcal{P}(M), W_2)$ joining μ_0 to μ_1 . By [4, Proposition 5.4], for each $t \in (0, 1)$, μ_t is absolutely continuous w.r.t. the volume measure m on M . Hence, we have $\mu_t \in \mathcal{P}_2(M)$, $t \in (0, 1)$.

The entropy is defined as a function on $\mathcal{P}(M)$ by

$$\text{Ent}(\nu) := \int_M \frac{d\nu}{dm} \log \left(\frac{d\nu}{dm} \right) dm(x)$$

if ν is absolutely continuous w.r.t. volume measure m on M and

$$\int_M \max \left\{ \log \left(\frac{d\nu}{dm} \right), 0 \right\} d\nu(x) < \infty;$$

otherwise, $\text{Ent}(\nu) := +\infty$.

Our main result of this work is:

Theorem 1.1. *Let Ric_x be the Ricci curvature at point $x \in M$. Let $x \mapsto K_x$ be a continuous function on M . For $D \subset M$, denote by $K(D) = \sup_{z \in D} K_z$. Let $B_x(r)$ denote the open ball of radius $r > 0$ centered at $x \in M$. The following properties are equivalent:*

- (i) $\text{Ric}_x \geq -K_x, \quad \forall x \in M.$
- (ii) *For each pair of $\mu_0, \mu_1 \in \mathcal{P}_2(M)$, it holds: $\forall t \in [0, 1]$,*

$$(1.1) \quad \begin{aligned} \text{Ent}(\mu_t) &\leq (1-t)\text{Ent}(\mu_0) + t\text{Ent}(\mu_1) \\ &\quad + \frac{t(1-t)}{2} \int_M K(B_x(\rho(F(x), x))) \rho^2(F(x), x) \mu_0(dx), \end{aligned}$$

where the map $F : M \rightarrow M$ is the unique optimal transport map between μ_0 and μ_1 for the L^2 -Wasserstein distance, which can be expressed by $F(x) = \exp_x(-\nabla \varphi)$ for some function φ . Here $\mu_t = (F_t)_* \mu_0$, $t \in [0, 1]$, is the geodesic joining μ_0 to μ_1 , where $F_t(x) := \exp_x(-t \nabla \varphi(x))$ for $t \in [0, 1]$.

As an application of this result, we can obtain the following volume growth estimate when the Ricci curvature of Riemannian manifold is locally lower bounded.

Proposition 1.2. *Assume $\text{Ric}_x \geq -K_x$ for every $x \in M$. For fixed $x_0 \in M$, consider the volume $V_R := m(\bar{B}_R(x_0))$ of the closed ball centered at x_0 with diameter R . Then for all $R \geq 2\varepsilon > 0$,*

$$(1.2) \quad V_R \leq V_{2\varepsilon} \left(\frac{V_{2\varepsilon}}{V_\varepsilon} \right)^{\frac{R}{\varepsilon}} \exp \left[\frac{K(B_{x_0}(R+2\varepsilon))}{2} R(R+\varepsilon) \right].$$

In particular, if $\text{Ric}_x \geq -C(1 + \rho_o(x))$, $\forall x \in M$, for some constant $C > 0$, then

$$(1.3) \quad V_R \leq V_{2\varepsilon} \left(\frac{V_{2\varepsilon}}{V_\varepsilon} \right)^{\frac{R}{\varepsilon}} \exp \left[\frac{C(1 + \rho_o(x_0) + R + 2\varepsilon)}{2} R(R+\varepsilon) \right].$$

According to [2], the local curvature bound $\text{Ric}_x \geq -K_x$ can also be characterized by the log-Harnack inequality and gradient estimate of the heat semigroup. To be more precise, let $p_t(x, y)$ be the heat kernel on M , that is, the minimal positive functional solution of the heat equation $(\frac{1}{2}\Delta - \frac{\partial}{\partial t})p_t(x, y) = 0$. Let $(W_t(x))$ be the Brownian motion on M with starting point x and life time $\zeta(x)$, where $\zeta(x) = \lim_{N \rightarrow \infty} \zeta_N(x)$, and

$$\zeta_N(x) := \inf\{t > 0; \rho(W_t(x), o) \geq N\}, \quad N \geq 1.$$

The associated semigroup is given by

$$(1.4) \quad P_t f(x) = \mathbb{E}[f(W_t(x)) \mathbf{1}_{t < \zeta(x)}], \quad t \geq 0,$$

for $f \in \mathcal{B}_b(M)$, where $\mathcal{B}_b(M)$ stands for the set of all bounded measurable functions on M . For any $D \subset M$, let

$$D_r = \{z \in M; \rho(z, D) \leq r\}, \quad r > 0.$$

For a given bounded open domain $D \subset M$, set

$$\mathcal{C}_D = \{\phi \in C^2(\bar{D}); \phi|_D > 0, \phi|_{\partial D} = 0\}.$$

Proposition 1.3 ([2] Theorem 1.1). *The following statements are equivalent:*

- (i) $\text{Ric}_x \geq -K_x, \quad \forall x \in M$.
- (iii) *For any bounded open domain $D \subset M$ and $\phi \in \mathcal{C}_D$, the log-Harnack inequality*

$$\begin{aligned} & P_t \log f(y) - \log \left(P_t f(x) - 1 - P_t 1(x) \right) \\ & \leq \frac{\rho(x, y)^2}{2} \left(\frac{K(D_{\rho(x, y)})}{1 - e^{-2K(D_{\rho(x, y)})t}} + \frac{c_D(\phi)^2 (e^{2K(D_{\rho(x, y)})t} - 1)}{2K(D_{\rho(x, y)})\phi(y)^4} \right), \\ & t > 0, \quad y \in D, \quad x \in M, \end{aligned}$$

holds for strictly positive $f \in \mathcal{B}_b(M)$, where

$$c_D(\phi) = \sup_D \{5|\nabla \phi|^2 - \frac{1}{2}\phi(\Delta \phi)\} \in [0, \infty).$$

(iv) For any bounded open domain $D \subset M$ and any $\phi \in \mathcal{C}_D$,

$$|\nabla P_t f|^2(x) \leq \{P_t f^2 - (P_t f)^2\}(x) \left(\frac{K(D)}{1 - e^{-2K(D)t}} + \frac{c_D(\phi)^2(e^{2K(D)t} - 1)}{2K(D)\phi(x)^4} \right)$$

holds for all $x \in D$, $t > 0$, $f \in \mathcal{B}_b(M)$. If moreover $P_t 1 = 1$, then the statements above are also equivalent to:

(v) For any bounded open domain $D \subset M$ and any $\phi \in \mathcal{C}_D$, the Harnack inequality

$$P_t f(y) \leq P_t f(x) + \rho(x, y) \left(\frac{K(D)}{1 - e^{-2K(D)t}} + \frac{c_D(\phi)^2(e^{2K(D)t} - 1)}{2K(D) \inf_{\ell(x, y)} \phi^4} \right)^{\frac{1}{2}} (P_t f^2(y))^{\frac{1}{2}}$$

holds for non-negative $f \in \mathcal{B}_b(M)$, $t > 0$ and $x, y \in D$ such that the minimal geodesic $\ell(x, y)$ linking x with y is contained in D .

2 Proof of the results

This section is devoted to the proof of our results. The volume distortion coefficient v_t plays a crucial role in our analysis. We recall the definition and some basic properties of v_t from [4]. For $t \in [0, 1]$ and $x, y \in M$, let

$$Z_t(x, y) = \{z \in M; \rho(x, z) = t\rho(x, y) \text{ and } \rho(z, y) = (1 - t)\rho(x, y)\}.$$

$Z_t(x, y)$ is the barycenter between x and y . For a set $Y \subset M$, define

$$Z_t(x, Y) = \bigcup_{y \in Y} Z_t(x, y).$$

Letting $B_r(y) \subset M$ denote the open ball of radius $r > 0$ centered at $y \in M$, for $t \in (0, 1]$ the volume distortion coefficient v_t is defined by

$$(2.1) \quad v_t(x, y) = \lim_{r \rightarrow 0} \frac{m(Z_t(x, B_r(y)))}{m(B_{tr}(y))}.$$

It always holds $v_t(x, y) > 0$ and $v_1(x, y) = 1$. We recall the following comparison bound on volume distortion from [4, Corollary 2.2]

Lemma 2.1 ([4]). *Assume that $\text{Ric} \geq (n - 1)k$ throughout M for some $k \in \mathbb{R}$. Then for $x, y \in M$ with $y \notin \text{cut}(x)$ and $t \in (0, 1)$,*

$$v_t(x, y) \geq \left(\frac{S(t\rho(x, y); k)}{S(\rho(x, y); k)} \right)^{n-1},$$

where $S(r; k)$ is defined by

$$(2.2) \quad S(r; k) = \begin{cases} \sin\left(r\sqrt{\frac{k}{n-1}}\right) / \left(r\sqrt{\frac{k}{n-1}}\right), & k > 0, \\ 1, & k = 0, \\ \sinh\left(r\sqrt{\frac{-k}{n-1}}\right) / \left(r\sqrt{\frac{-k}{n-1}}\right), & k < 0. \end{cases}$$

Proof of Theorem 1.1:

(i) \Rightarrow (ii): As $\mu_t = (F_t)_*\mu_0$, we have for any bounded continuous function $g(\cdot)$ on M ,

$$\int_M g(x) d\mu_t(x) = \int_M g(F_t(x)) d\mu_0(x),$$

which implies

$$\int_M g(x) \frac{d\mu_t}{dm}(x) dm(x) = \int_M g(F_t(x)) \frac{d\mu_0}{dm}(x) dm(x).$$

By changing of variable, we get

$$\int_M g(F_t(x)) \frac{d\mu_t}{dm}(F_t(x)) \det(dF_t(x)) dm(x) = \int_M g(F_t(x)) \frac{d\mu_0}{dm}(x) dm(x).$$

By the arbitrariness of g , we get

$$\frac{d\mu_t}{dm}(F_t(x)) \det(dF_t(x)) = \frac{d\mu_0}{dm}(x).$$

Let $J_t(x) = \det(dF_t(x))$. It holds

$$\begin{aligned} \text{Ent}(\mu_t) &= \int_M \log \frac{d\mu_t}{dm}(x) d\mu_t(x) \\ &= \int_M \log \frac{d\mu_t}{dm}(F_t(x)) d\mu_0(x) \\ &= \int_M \left(\log \frac{d\mu_0}{dm}(x) - \log J_t(x) \right) d\mu_0(x) \\ &= \text{Ent}(\mu_0) - \int_M \log J_t(x) d\mu_0(x). \end{aligned}$$

According to [4, Theorem 4.2], $F(x)$ does not belong to the cut locus of x for μ almost everywhere x . Then by Lemma 6.1 of [4], $J_t = \det(dF_t(x))$ satisfies the inequality

$$(2.3) \quad J_t^{1/n}(x) \geq (1-t)[v_{1-t}(F(x), x)]^{1/n} + t[v_t(x, F(x))]^{1/n} J_1^{1/n}(x).$$

Therefore,

$$\begin{aligned} & -\text{Ent}(\mu_t) + (1-t)\text{Ent}(\mu_0) + t\text{Ent}(\mu_1) \\ &= \int_M \log J_t(x) \mu_0(dx) - t \int_M \log J_1(x) \mu_0(dx) \\ (2.4) \quad & \geq n \int_M \log [(1-t)v_{1-t}(x, F(x))^{\frac{1}{n}} + tv_t(x, F(x))^{\frac{1}{n}} J_1(x)^{\frac{1}{n}}] \mu_0(dx) \\ & \quad - t \int_M \log J_1(x) \mu_0(dx). \end{aligned}$$

In order to use the comparison bound on volume distortion ([4, Corollary 2.2]), for each $x \in \text{supp}\mu_0$, we consider the open set $B_x(\rho(x, F(x)) + \varepsilon)$, which is a geodesic ball centered

at x with diameter $\rho(x, F(x)) + \varepsilon$, where $0 < \varepsilon < 1$. We look on $B_x(\rho(x, F(x)) + \varepsilon)$ as a manifold, which satisfies

$$\begin{aligned}\text{Ric}_z &\geq -K_z \\ &\geq -K(B_x(\rho(x, F(x)) + \varepsilon)), \quad \forall z \in B_x(\rho(x, F(x)) + \varepsilon).\end{aligned}$$

Then by noting that the lower bound of Ricci curvature is negative, Lemma 2.1 implies

$$\begin{aligned}(2.5) \quad &\log((1-t)v_{1-t}(x, F(x))^{\frac{1}{n}} + tv_t(x, F(x))^{\frac{1}{n}} J_1(x)^{\frac{1}{n}}) \\ &\geq \log\left((1-t) \left[\frac{S((1-t)\rho(x, F(x)); -K(B_x(\rho(x, F(x)) + \varepsilon)))}{S(\rho(x, F(x)); -K(B_x(\rho(x, F(x)) + \varepsilon)))} \right]^{1-\frac{1}{n}} \right. \\ &\quad \left. + t \left[\frac{S(t\rho(x, F(x)); -K(B_x(\rho(x, F(x)) + \varepsilon)))}{S(\rho(x, F(x)); -K(B_x(\rho(x, F(x)) + \varepsilon)))} \right]^{1-\frac{1}{n}} J_1(x)^{\frac{1}{n}} \right).\end{aligned}$$

Therefore, combining (2.5) with (2.4) and letting $\varepsilon \rightarrow 0$, we get

$$\begin{aligned}& -\text{Ent}(\mu_t) + (1-t)\text{Ent}(\mu_0) + t\text{Ent}(\mu_1) \\ & \geq n \int_M \log \left[(1-t) \left[\frac{S((1-t)\rho(x, F(x)); -K(B_x(\rho(x, F(x))))}{S(\rho(x, F(x)); -K(B_x(\rho(x, F(x))))} \right]^{1-\frac{1}{n}} \right. \\ & \quad \left. + t \left[\frac{S(t\rho(x, F(x)); -K(B_x(\rho(x, F(x))))}{S(\rho(x, F(x)); -K(B_x(\rho(x, F(x))))} \right]^{1-\frac{1}{n}} J_1(x)^{\frac{1}{n}} \right] \mu_0(dx) - t \int_M \log J_1(x) \mu_0(dx) \\ & \geq (n-1) \int_M [(1-t) \log S((1-t)\rho(x, F(x)); -K(B_x(\rho(x, F(x)))) \\ & \quad + t \log S(t\rho(x, F(x)); -K(B_x(\rho(x, F(x)))) - S(\rho(x, F(x)); -K(B_x(\rho(x, F(x)))))] \mu_0(dx) \\ & \geq -\frac{t(1-t)}{2} \int_M K(B_x(\rho(x, F(x)))) \rho^2(x, F(x)) \mu_0(dx),\end{aligned}$$

where we have used the concavity of the logarithm and following inequality (cf. [9])

$$(1-t) \log S((1-t)r; k) + t \log S(tr; k) - \log S(r; k) - \frac{t(1-t)}{2} \frac{k}{n-1} r^2 \geq 0.$$

(ii) \Rightarrow (i): If (i) does not hold, then there exists a point $z_0 \in M$ such that $\text{Ric}_{z_0} < -K_{z_0}$. By the continuous property of K_x , there exist two positive constants ε_0 and δ with $\varepsilon_0, \delta < 1$ such that

$$\text{Ric}_z < -K_{z_0} - \varepsilon_0, \quad \forall z \in B_\delta(z_0),$$

and $B_\delta(z_0)$ is geodesically complete. Next, similar to [9], we can construct two probability measures μ_0, μ_1 with $\text{supp}(\mu_0) \subset B_\delta(z_0), \text{supp}(\mu_1) \subset B_\delta(z_0)$ such that

$$(2.6) \quad \text{Ent}(\mu_{1/2}) - \frac{1}{2}\text{Ent}(\mu_0) - \frac{1}{2}\text{Ent}(\mu_1) \geq \frac{K_{z_0} + \varepsilon_0/2}{8} W_2^2(\mu_0, \mu_1).$$

Indeed, let e_1, e_2, \dots, e_n be an orthonormal basis of $T_{z_0}M$ such that

$$R(e_1, e_i)e_1 = k_i e_i \quad \text{for some numbers } k_i, i = 1, \dots, n.$$

Then $\sum_{i=1}^n k_i = \text{Ric}_{z_0}(e_1, e_1) \leq -K_{z_0} - \varepsilon_0$. For $\beta, r > 0$, let $A_1 = B_\beta(\exp_{z_0}(re_1))$, $A_0 = B_\beta(\exp_{z_0}(-re_1))$ be geodesic balls and

$$A_{1/2} = \exp \left(\left\{ y \in T_{z_0}M : \sum_{i=1}^n (y_i/\beta_i)^2 \leq 1 \right\} \right)$$

with $\beta_i = \beta(1 + r^2(k_i + \frac{\varepsilon_0}{2n})/2)$. By choosing $\beta \ll r \ll \delta$, one gets that $\gamma_{1/2} \in A_{1/2}$ for each minimizing geodesic $\gamma : [0, 1] \rightarrow M$ with $\gamma_0 \in A_0$, $\gamma_1 \in A_1$. Let μ_0 and μ_1 be the normalized uniform distribution in A_0 and A_1 respectively, and let ν be the normalized uniform distribution in $A_{1/2}$. Then

$$\text{Ent}(\mu_0) = \text{Ent}(\mu_1) = -\log m(A_0) = -\log c_n - n \log \beta + O(\beta^2),$$

where $c_n = m(B_1)$ in \mathbb{R}^n , and

$$\begin{aligned} \text{Ent}(\nu) &= -\log m(A_{1/2}) = -\log c_n - \sum_{i=1}^n \log \beta_i + O(\beta^2) \\ &= -\log c_n - n \log \beta - r^2(\varepsilon_0/2 + \sum_{i=1}^n k_i)/2 + O(r^4) + O(\beta^2) \\ &\geq -\log c_n - n \log \beta + \frac{r^2(K_{z_0} + \varepsilon_0/2)}{2} + O(r^4) + O(\beta^2) \end{aligned}$$

Since the support of $\mu_{1/2}$ is contained in the set $A_{1/2}$, one gets

$$\text{Ent}(\mu_{1/2}) \geq \text{Ent}(\nu).$$

Consequently,

$$\begin{aligned} (2.7) \quad & \text{Ent}(\mu_{1/2}) - \frac{1}{2}\text{Ent}(\mu_0) - \frac{1}{2}\text{Ent}(\mu_1) \\ & \geq \frac{r^2(K_{z_0} + \varepsilon_0/2)}{2} + O(r^4) + O(\beta^2) \\ & \geq \frac{K_{z_0} + \varepsilon_0/2}{8} W_2(\mu_0, \mu_1)^2 \end{aligned}$$

for $\beta \ll r \ll \delta$. Hence, we obtain (2.6).

On the other hand, since $\text{supp}(\mu_0)$ and $\text{supp}(\mu_1)$ are contained in $B_{r+\beta}(z_0)$, one gets that for each $x \in \text{supp}\mu_0$, $\rho(x, F(x)) \leq 2(r + \beta)$. By (ii), we have

$$\begin{aligned} (2.8) \quad & \text{Ent}(\mu_{1/2}) - \frac{1}{2}\text{Ent}(\mu_0) - \frac{1}{2}\text{Ent}(\mu_1) \\ & \leq \frac{1}{8} \int_{A_0} K(B_x(\rho(x, F(x)))) \rho^2(x, F(x)) \mu_0(dx) \\ & \leq \frac{1}{8} K(B_{z_0}(3\beta + 3r)) W_2^2(\mu_0, \mu_1) \leq \frac{1}{8} K(B_{z_0}(6r)) W_2^2(\mu_0, \mu_1). \end{aligned}$$

Since $\lim_{r \rightarrow 0} K(B_{z_0}(6r)) = K_{z_0} < K_{z_0} + \varepsilon_0/2$, we can choose $r > 0$ small enough so that

$$\frac{K(B_{z_0}(6r))}{8} < \frac{K_{z_0} + \varepsilon_0/2}{8}.$$

Thus, by (2.8) we get

$$\text{Ent}(\mu_{1/2}) - \frac{1}{2}\text{Ent}(\mu_0) - \frac{1}{2}\text{Ent}(\mu_1) < \frac{K_{z_0} + \varepsilon_0/2}{8} W_2^2(\mu_0, \mu_1),$$

which is in contradiction with (2.6). We complete the proof of this theorem. \square

Next, we shall show how to use our characterization of lower bound of Ricci curvature through convexity of relative entropy to study the volume growth property of Riemannian manifold M .

Proof of Proposition 1.2: Let μ_0 and μ_1 be the uniform distribution on $\bar{B}_\varepsilon(x_0)$ and $\bar{B}_R(x_0)$ respectively, i.e.

$$\mu_0(dx) = \frac{\mathbf{1}_{\bar{B}_\varepsilon(x_0)}}{m(\bar{B}_\varepsilon(x_0))}, \quad \mu_1(dx) = \frac{\mathbf{1}_{\bar{B}_R(x_0)}}{m(\bar{B}_R(x_0))}.$$

Let $(\mu_t)_{t \in [0,1]}$ be the geodesic in $\mathcal{P}_2(M)$ connecting μ_0 and μ_1 . Then, according to Theorem 1.1,

$$\begin{aligned} \text{Ent}(\mu_t) &\leq (1-t)\text{Ent}(\mu_0) + t\text{Ent}(\mu_1) \\ &\quad + \frac{t(1-t)}{2} \int_M K(B_x(\rho(x, F(x)))) \rho^2(x, F(x)) \mu_0(dx). \end{aligned}$$

Since $\mu_t = (F_t)_* \mu_0$, we know that $\text{supp } \mu_t \subset \bar{B}_{\varepsilon+t(R+\varepsilon)}(x_0)$. By Jensen's inequality,

$$\text{Ent}(\mu_t) \geq \text{Ent}\left(\frac{\mathbf{1}_{\bar{B}_{\varepsilon+t(R+\varepsilon)}}}{m(\bar{B}_{\varepsilon+t(R+\varepsilon)})}\right) = -\log V_{\varepsilon+t(R+\varepsilon)}.$$

Hence,

$$\begin{aligned} -\log V_{\varepsilon+t(R+\varepsilon)} &\leq (1-t)\text{Ent}(\mu_0) + t\text{Ent}(\mu_1) \\ &\quad + \frac{t(1-t)}{2} \int_M K(B_x(\rho(x, F(x)))) \rho(x, F(x))^2 d\mu_0(x) \\ &\leq -(1-t)\log V_\varepsilon - t\log V_R + \frac{t(1-t)}{2} K(B_{x_0}(R+2\varepsilon))(R+\varepsilon)^2. \end{aligned}$$

Take $t = \varepsilon/(R+\varepsilon)$, then $\varepsilon+t(R+\varepsilon) = 2\varepsilon$, and the desired inequality (1.2) follows immediately from previous inequality. \square

Acknowledgments This work is supported by NSFC (No.11301030, 11371099), 985-project, Specialized Research Fund for the Doctoral Program of Higher Education of China (No. 20120071120001).

References

- [1] L. Ambrosio, N. Gigli, G. Savaré, Gradient Flows in Metric Spaces and in the Space of Probability Measures. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2005.
- [2] M. Arnaudon, A. Thalmaier, F.-Y. Wang, Equivalent Harnack and gradient inequalities for pointwise curvature lower bound. Bull. Sci. math., 138 (2014), 643-655.
- [3] I. Chavel, Riemannian geometry: a modern introduction, Cambridge University Press, 1993.
- [4] Dario Cordero-Erausquin, R. McCann, M. Schmuckenschläger, A Riemannian interpolation inequality à la Borel, Brascamb and Lieb. Invent. Math., 146 (2001), 219-257.
- [5] R. Jordan, D. Kinderlehrer, F. Otto, The variational formulation of the Fokker-Planck equation. SIAM J. Math. Anal. 29 (1998), 1-17.
- [6] J. Lott, Manifolds with quadratic curvature decay and fast volume growth. Math. Ann. 325 (2003), 525-541.
- [7] J. Lott, C. Villani, Ricci curvature for metric-measure spaces via optimal transport, Ann. Math. 169 (3) (2009), 903-991.
- [8] R.J. McCann, Polar factorization of maps on Riemannian manifolds, Geom. Funct. Anal. 11 (2001), 589-608.
- [9] K.T. Sturm, von Renesse, Transport inequalities, gradient estimates, entropy, and Ricci curvature. Comm. Pure Appl. Math. 58 (2005), 923-940.
- [10] K.T. Sturm, Convex functionals of probability measures and nonlinear diffusions on manifolds, J. Math. Pures Appl. 84 (2005), 149-297.
- [11] K.T. Sturm, On the geometry of metric measure spaces. I, Acta Math., 196 (1) (2006), 65-131.
- [12] K.T. Sturm, On the geometry of metric measure spaces. II, Acta Math., 196 (1) (2006), 133-177.
- [13] C. Villani, Optimal Transport, Old and New, Grundlehren Math. Wiss., vol. 338, Springer, Berlin-Heidelberg, 2009.